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# Checking for optimal solutions in some $NP$ -complete problems

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For some weighted  $NP$ -complete problems, checking whether a proposed solution is optimal is a non-trivial task. Such is the case for the celebrated traveling salesman problem, or the spin-glass problem in 3 dimensions. In this letter, we consider the weighted tripartite matching problem, a well known  $NP$ -complete problem. We write mean-field finite temperature equations for this model, and show that they become exact at zero temperature. As a consequence, given a possible solution, we propose an algorithm which allows to check in a polynomial time if the solution is indeed optimal. This algorithm is generalized to a class of variants of the multiple traveling salesmen problem.

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A combinatorial optimization problem is defined as the minimization of a cost function over a discrete set of configurations [1]. Typically, in statistical physics, finding the ground state of an Ising spin-glass (Ising model with random interactions) is a combinatorial optimization problem, where the cost function is the magnetic energy of the system. The size  $N$  of an optimization problem is the number of degrees of freedom of the system over which the minimization is performed. In a spin-glass problem, the size  $N$  is just the number of spins. In the following, we shall be interested only in problems for which the cost function can be calculated in a time which is polynomial in the size  $N$ . This very wide class of problems is called the  $NP$  class. Many physical problems belong to this class.

An optimisation problem  $Q$  is said to be  $NP$ -complete if it is in  $NP$ , and if all problems in  $NP$  can be shown to be polynomially algorithmically reducible to  $Q$ . Therefore,  $NP$ -complete problems are in some sense the most difficult  $NP$  problems. All  $NP$ -complete problems are algorithmically polynomially equivalent. The archetype of such problems is the celebrated traveling salesman problem (TSP): Given  $N$  cities, find the shortest path going through each city once and only once. Although many algorithms exist which provide exact solutions for small enough  $N$  or almost optimal solutions for larger  $N$ , there is no known algorithm which provides the exact shortest path in a polynomial time. This is due to the combinatorial complexity of the paths, and to the strong non-convexity of the cost function in phase space, which manifests itself by an exponentially large number of local minima.

As stated above, any combinatorial optimization problem which is algorithmically equivalent to the TSP is  $NP$ -complete. Famous examples of such problems include the spin-glass (SG) problem in dimension  $d \geq 3$  [2], the Hamiltonian path (HP) problem (given a graph, find a path which goes through each points of the graph once and only once), the weighted tripartite matching problem (TMP), 3-satisfiability, etc. For an extensive list of  $NP$ -complete problems, see ref. [3].

In the spin-glass problem, physicists and computer scientists have developed algorithms which allow to com-

pute the exact ground state for small enough systems [4]. The analysis of these exact ground states and low-lying excited states allows to check the validity of various spin-glass theories [5].

If there is a polynomial time algorithm to solve a combinatorial optimization problem, the problem is said to belong to the  $P$ -class. Many problems belong to  $P$ , among which the assignment problem (also known as the weighted bipartite matching problem denoted BMP), the spin-glass problem in  $d = 2$ , etc. Of course the  $P$  class is included in the  $NP$  class, but it is not known if the inclusion is strict. This is the celebrated “Is  $P = NP$  ?” problem.

The strong analogy between combinatorial optimization problems and the physics of disordered systems was recognized in the early 1980s [6], and was the basis of the development of simulated annealing techniques in optimization. At finite temperatures, many of these optimization problems exhibit glassy behaviour as seen in disordered systems [7, 8, 9, 11, 12, 13].

As mentioned above, when going to larger sizes, finding exact ground states of  $NP$ -complete problems becomes impossible, and one resorts to non-deterministic methods like Monte Carlo algorithms or Markov chains. These methods provide candidates for solutions to the optimization problem. In some cases, checking that one has a true solution of the optimization problem is an easy task. For instance, in the HP problem, given a path, it is easy to check whether this path is Hamiltonian and thus solves the problem. In many weighted problems however, checking that one has a solution is a very non trivial task. Such is the case for example in the TSP problem. Given a tour, there is no known algorithm to determine whether it is the optimal path, except by actually computing the optimal path. Similarly, given a set of coupling  $J_{ij}$  and a spin configuration  $S_i$ , there is no known algorithm to check that this configuration is the ground state of the spin-glass, except by computing its actual ground state.

In the present paper, we study the weighted multipartite matching problem. To simplify, we specialize to the tripartite case. Before defining it, we first recall the assignment problem or BMP (also called the wedding problem). Assume we have two sets  $\{i = 1, \dots, N\}$  and

$\{j = 1, \dots, N\}$ . A positive pairing cost  $l_{ij}$  is assigned to each matching of  $i$  and  $j$ . Note that the matrix  $l_{ij}$  has no reason to be taken symmetric. A matching of the two sets is in fact a permutation  $P$  of the set  $\{j\}$  and the corresponding cost is

$$L = \sum_{i=1}^N l_{iP(i)} \quad (1)$$

Solving the BMP amounts to finding the permutation  $P$  which minimizes (1), i.e. the complete pairing of  $\{i\}$  and  $\{j\}$  which has lowest cost.

The simplest case of multipartite matching problem is the tripartite one. The TMP involves three sets  $\{i = 1, \dots, N\}$ ,  $\{j = 1, \dots, N\}$ ,  $\{k = 1, \dots, N\}$  and a positive cost function  $l_{ijk}$ . A matching of the 3 sets is defined by a permutation  $P$  of the set  $\{j\}$  and a permutation  $Q$  of the set  $\{k\}$  with cost

$$L = \sum_{i=1}^N l_{iP(i)Q(i)} \quad (2)$$

Solving the TMP amounts to finding the permutations  $P$  and  $Q$  which minimize (2). Generalization to multipartite matchings is straightforward. As mentioned above, the bipartite version is polynomial, whereas the tri-, quadri-, etc. matching are  $NP$ -complete. Note that in the BMP, the number of possible configurations is  $N!$  whereas it is  $(N!)^2$  in the TMP. The phase diagram of these multipartite matching problems has been studied recently for sets of random independent costs [13].

In this article, we first formulate the finite temperature TMP as an integral over complex fields. The saddle-point equations are derived and are shown to provide the exact solution to the TMP at zero temperature. Unfortunately, this does not help in solving the problem. However, from these equations, we find that given a tripartite assignment, one can check whether it is the absolute minimum of the cost function in a polynomial time. This method can easily be applied to a whole class of multiple traveling salesmen problems [14]. This of course does not violate the  $NP$ -complete character of these problems. We are not aware of any other *weighted*  $NP$ -complete problems which could be checked in a polynomial time, and hopefully this method could be generalized to other problems.

It is possible to give an integral formulation of the TMP using techniques similar to those used for the BMP [8, 9]. We want to calculate the partition function  $Z$  of the TMP at temperature  $T$ . Denoting by  $k_B$  the Boltzmann constant and  $\beta = 1/k_B T$ , we have

$$Z = \sum_{P, Q \in S_N} \exp \left( -\beta \sum_{i=1}^N l_{iP(i)Q(i)} \right) \quad (3)$$

where  $S_N$  denotes the group of permutations of  $N$  objects (symmetric group). We define

$$U_{ijk} = \exp(-\beta l_{ijk})$$

$$\begin{aligned} Z = & \int \prod_{i=1}^N \frac{d\phi_i d\phi_i^*}{\pi} \frac{d\psi_i d\psi_i^*}{\pi} \frac{d\chi_i d\chi_i^*}{\pi} \\ & \times \exp \left( - \sum (\phi_i \phi_i^* + \psi_i \psi_i^* + \chi_i \chi_i^*) \right) \\ & \times \exp \left( \sum_{i,j,k=1}^N U_{ijk} \phi_i \psi_j \chi_k \right) \prod_{i=1}^N (\phi_i^* \psi_i^* \chi_i^*) \end{aligned} \quad (4)$$

Let us show that this is an exact expression for the partition function of the TMP. We use Wick's theorem. We first note that the contraction of each field with its conjugate is just equal to 1. Expanding the exponential in powers of  $U_{ijk}$ , we should contract each  $\phi_i, \psi_i, \chi_i$  with its conjugate  $\phi_i^*, \psi_i^*, \chi_i^*$ . Since the integrand is linear in each  $\phi_i^*, \psi_i^*, \chi_i^*$ , the expansion in powers of  $U_{ijk}$  should be limited to first order. This obviously proves equation (4).

A similar formula can easily be obtained for the BMP, involving only two kinds of fields instead of three.

A standard way to approximate eq. (4) is to perform a saddle-point expansion. The saddle-point equations read

$$\begin{aligned} \frac{1}{\phi_i^*} &= \phi_i \\ \phi_i^* &= \sum_{j,k=1}^N U_{ijk} \psi_j \chi_k \end{aligned}$$

and similar equations for the other variables. Eliminating the conjugate variables, we have the  $3N$  equations

$$1 = \phi_i \sum_{j,k=1}^N U_{ijk} \psi_j \chi_k \quad (5)$$

for any  $i$  and the analogous equations for the other variables.

There are  $3N$  saddle-point equations for the  $3N$  variables  $\phi_i, \psi_i, \chi_i$ . These equations are complicated nonlinear equations and can be solved numerically. They differ from those obtained from the cavity method [10, 13]. However, they display some interesting properties in the zero temperature limit. Introducing new variables

$$\phi_i = e^{\beta a_i}, \psi_i = e^{\beta b_i}, \chi_i = e^{\beta c_i}$$

one can see that solving equations (5) in the zero temperature limit amounts to finding two permutations  $P$  and  $Q$  such that

$$a_i + b_{P(i)} + c_{Q(i)} = l_{iP(i)Q(i)} \quad (6)$$

for any  $i$ , and

$$a_i + b_j + c_k < l_{ijk} \quad (7)$$

for any triplet  $(i, j, k) \neq (i, P(i), Q(i))$ . The total cost of the matching is

$$L = \sum_{i=1}^N l_{iP(i)Q(i)} = \sum_{i=1}^N (a_i + b_i + c_i) \quad (8)$$

We now proceed to prove that these equations are exact, i.e. their solutions (if they exist) provide the optimal tripartite matching. We first note that if we have two sets of solutions  $(a, b, c)$  and  $(a', b', c')$  which satisfy equations (6) and (7) with the same  $P$  and  $Q$ , they necessarily have the same total cost, due to equations (6). Now consider another pair of permutations  $P'$  and  $Q'$ . The total cost  $L'$  associated to these permutations is

$$L' = \sum_{i=1}^N l_{iP'(i)Q'(i)}$$

According to eq. (7)

$$a_i + b_{P'(i)} + c_{Q'(i)} \leq l_{iP'(i)Q'(i)}$$

and therefore, summing over  $i$  implies

$$L \leq L'$$

Therefore, any matching other than  $(P, Q)$  have larger cost, which proves that  $(P, Q)$  generates the optimal matching.

A formulation similar to (6) and (7) has been known for the BMP, with only two sets of variables  $a$  and  $b$ . It can be shown that these equations can be solved using the so-called Hungarian method [15] which is an  $O(N^3)$  algorithm.

In the case of the TMP, these equations unfortunately cannot be solved in a polynomial time. However, as we shall now see, they allow to check in a polynomial time whether a proposed solution is the actual optimal matching.

Let us consider the reciprocal problem: Assume we are given a matching. How can we check whether it is optimal?

The matching is defined by a set of  $N$  costs  $\{l_{iP(i)Q(i)}\}$ .

If this set is optimal, then there exists a set of  $3N$  variables  $a_i, b_i, c_i$  such that the constraints (6) and (7) would be satisfied.

These are a set of  $N$  equations and  $N^3 - N$  linear inequalities for  $3N$  variables. We can simplify further by using equations (6), and obtain a set of  $N^3 - N$  inequalities for the  $2N$  variables  $a_i$  and  $b_i$

$$a_i + b_j - a_{Q^{-1}(k)} - b_{PQ^{-1}(k)} \leq l_{ijk} - l_{Q^{-1}(k)PQ^{-1}(k)k} \quad (9)$$

This set of linear inequalities with integer coefficients belongs to the well known class of optimization problems

called "linear inequalities" [1]. This problem, which is related to linear programming, is known to be solvable in a time which is polynomial in the size  $2N$  of the problem [16, 17]. Therefore, we can find, in a polynomial time, i) either that there is a solution to (9), in which case the proposed solution is the optimal solution to the TMP, ii) or that there is no solution to these inequalities, in which case the proposed solution is not the optimal one.

We have not found a general proof of existence of the  $a_i, b_i, c_i$  for the optimal solution.

Let us show how this method can be generalized to some variants of the multiple traveling salesmen problem (MTSP). The MTSP is similar to the TSP, except that the number of travelers is not equal to one. Consider a set of  $N$  points (in an abstract space) with positive distances  $l_{ij}$ . We consider the TMP in which

$$l_{i,j,k} = \frac{1}{2}(l_{ij} + l_{jk})$$

if  $(i, j, k)$  are distinct

$$l_{i,j,k} = \infty$$

if 2 indices are equal.

With this choice, a finite cost tripartite assignment can be viewed as a set of loops visiting each point once and only once. The corresponding TMP thus amounts to a MTSP, with any number of salesmen, each visiting at least 3 cities. If we are presented with a possible optimal path, say  $\{l_{12}, l_{23}, \dots, l_{N1}\}$ , we look for two sets of variables  $a_i$  and  $c_i$  which satisfy the linear inequalities:

$$a_i + c_k - a_{j-1} - c_{j+1} \leq \frac{1}{2}(l_{ij} + l_{jk} - l_{j-1,j} - l_{j,j+1})$$

Again, although the problem is  $NP$ -complete, checking the optimality of the solution can be achieved in a polynomial time. The whole method can be easily generalized to MTSP with loop sizes greater than  $p$  in terms of  $p$ -partite weighted matching problems, to find criteria of optimality which are polynomial.

We have proposed an integral representation of the TMP, from which we derive some mean-field equations. These equations turn out to be exact at zero temperature, and could have in fact been derived directly without going through the mean-field method. However, we find this approach interesting, as it might be generalized to other  $NP$ -complete problems. The zero temperature equations cannot be solved in a polynomial time. However, given a test solution to the problem, they allow to check in a polynomial time whether the proposed solution is indeed the optimum of problem. This situation is quite unique in  $NP$ -complete problems, since in principle, for such problems, the existence of an exponentially large number of local minima prevents checking the optimality in a polynomial time. A generalization of this method to the TSP or to the spin-glass problem would be of great interest for the physics of disordered systems and is currently under investigation.

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